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Application of He's variational iteration method and Adomian's decomposition method to the fractional KdV–Burgers–Kuramoto equation

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ABSTRACT

In this work, the fractional KdV–Burgers–Kuramoto equation is studied. He's variational iteration method (VIM) and Adomian's decomposition method (ADM) are applied to obtain its solution. Comparison with HAM is made to highlight the significant features of the employed methods and their capability of handling completely integrable equations.

1. Introduction

In the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equations. In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering [1]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by fractional differential equations [2–4]. The solution of fractional differential equation is much involved. In general, there exists no method that yields an exact solution for fractional differential equations. Only approximate solutions can be derived using linearization or perturbation methods. This paper presents the analytical solutions of the fractional KdV–Burgers–Kuramoto equation by the variational iteration method (VIM) [5–17] and the Adomian decomposition method (ADM) [18–26]. The results obtained by using these methods are also compared to the results of the homotopy analysis method (HAM) in Ref. [27].

The fractional KdV–Burgers–Kuramoto (KBK) equation with space-fractional derivatives of the form

$$\frac{\partial u}{\partial t}+u\frac{\partial^{\alpha} u}{\partial x^{\alpha}}+a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^3 u}{\partial x^3}+c\frac{\partial^4 u}{\partial x^4}=0,\quad t>0,0<\alpha\leq 1, \tag{1}$$

where a, b, c are constants and α is a parameter describing the order of the fractional space-derivatives. The function $u(x,t)$ is assumed to be a causal function of space, i.e. vanishing for $t<0$ and $x<0$. The fractional derivatives are considered in Caputo's sense [28]. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha=1$, Eq. (1) reduces to the classical nonlinear KBK equation. More important, the above procedure is just an algebraic algorithm and can be easily applied in the symbolic computation system Maple.

Although there are a lot of studies for the classical KBK equation and some profound results have been established, it seems that detailed studies of the nonlinear fractional differential equation are only beginning.

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2. Preliminaries and notations

In this section, let us recall essentials of fractional calculus first. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. There are many books [1–4] that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grünwald–Letnikov's definition, Riemann–Liouville's definition, Caputo's definition and generalized function approach. For the purpose of this paper Caputo's definition of fractional differentiation will be used, taking the advantage of Gaputo's approach that the initial conditions for fractional differential equation with Caputo's derivatives take on the traditional form, similar to those for integer-order differential equation.

Definition 2.1. Caputo's definition of the fractional-order derivative is defined as

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}\tau}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau \quad (n-1<\mathrm{Re}(\alpha)\leq n, n\in N), \tag{2}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex, a is the initial value of function f.

In this paper only real and positive α will be considered. For Caputo's derivative we have

$$D^{\alpha}C=0 \quad (C \text{ is a constant}), \tag{3}$$

$$D^{\alpha}t^{\beta}=\begin{cases}0 & (\beta\leq\alpha-1),\\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha} & (\beta>\alpha-1).\end{cases} \tag{4}$$

Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation:

$$D^{\alpha}(\lambda f(t)+\mu g(t))=\lambda D^{\alpha}f(t)+\mu D^{\alpha}g(t), \tag{5}$$

where λ, μ are constants, and satisfies the so-called Leibnitz rule:

$$D^{\alpha}(g(t)f(t))=\sum_{k=0}^{\infty}\binom{\alpha}{k}g^{(k)}(t)D^{\alpha-k}f(t), \tag{6}$$

if $f(\tau)$ is continuous in $[a,t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[a,t]$.

In this paper, we consider Eq. (1), where the unknown function $u=u(x,t)$ is assumed to be a causal function of space, and the fractional derivative is taken in Caputo's sense as follows:

Definition 2.2. For n to be the smallest integer that exceeds α, Caputo's space-fractional derivative operator of order $\alpha>0$ is defined as

$$D_x^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x(x-\tau)^{n-\alpha-1}\frac{\partial^n u(\tau,t)}{\partial\tau^n}\mathrm{d}\tau & \text{if } n-1<\alpha<n\\ \dfrac{\partial^n u(x,t)}{\partial x^n} & \text{if } \alpha=n\in N.\end{cases} \tag{7}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult Ref. [4].

3. Basic idea of He's variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu+Nu=g(t), \tag{8}$$

where L is a linear operator, N a nonlinear operator and $g(t)$ an inhomogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t)=u_n(t)+\int_0^t\lambda\left(Lu_n(\tau)+N\tilde{u}_n(\tau)-g(\tau)\right)\mathrm{d}\tau, \tag{9}$$

where λ is a general Lagrangian multiplier which can be optimally identified via the variational theory. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation $\delta\tilde{u}_n=0$.

4. Basic idea of Adomian's decomposition method

We begin with the equation

$$Lu + R(u) + F(u) = g(t), \tag{10}$$

where L is the operator of the highest-ordered derivatives with respect to t and R is the remainder of the linear operator. The nonlinear term is represented by $F(u)$. Thus we get

$$Lu = g(t) - R(u) - F(u). \tag{11}$$

The inverse L^{-1} is assumed to be an integral operator given by

$$L_t^{-1} = \int_0^t (\cdot)\, \mathrm{d}t, \tag{12}$$

operating with the operator L^{-1} on both sides of Eq. (11) we have

$$u = f_0 + L^{-1}(g(t) - R(u) - F(u)), \tag{13}$$

where f_0 is the solution of homogeneous equation

$$Lu = 0, \tag{14}$$

involving the constants of integration. The integration constants involved in the solution of homogeneous equation (14) are to be determined by the initial or boundary condition, since the problem is an initial value or a boundary value problem.

The ADM assumes that the unknown function $u(x, t)$ can be expressed by an infinite series of the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \tag{15}$$

and the nonlinear operator $F(u)$ can be decomposed by an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n, \tag{16}$$

where $u_n(x, t)$ will be determined recurrently, and A_n are the so-called polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$An = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F\left(\sum_{n=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2 \ldots. \tag{17}$$

5. Application of VIM and ADM methods

We consider two different examples of the fractional KdV–Burgers–Kuramoto equation based on α values such as $\alpha = 1$ and $\alpha = 0.5$. VIM and ADM methods are applied to obtain analytical solutions of these examples. In the case of $\alpha = 1$, Eq. (1) reduces to the classical nonlinear KBK equation that can be seen below

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^3 u}{\partial x^3} + c\frac{\partial^4 u}{\partial x^4} = 0, \tag{18}$$

and for $\alpha = 0.5$, Eq. (1) will be in the form of Eq. (19)

$$\frac{\partial u}{\partial t} + u\frac{\partial^{0.5} u}{\partial x^{0.5}} + a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^3 u}{\partial x^3} + c\frac{\partial^4 u}{\partial x^4} = 0. \tag{19}$$

5.1. VIM implement for fractional KdV–Burgers–Kuramoto equation with $\alpha = 1$ (Eq. (18))

We first consider the application of VIM to Eq. (18) with the initial condition of:

$$u(x, 0) = kx^4, \quad k \in C. \tag{20}$$

Its correction variational functional in x and t can be expressed, respectively, as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \left\{\dot{u}_n + u_n u_n' + a u_n'' + b u_n''' + c u_n^{(4x)}\right\} \mathrm{d}\tau, \tag{21}$$

where prime indicates a differential with respect to x and dot denotes a differential with respect to t, λ is general Lagrangian multiplier.

After some calculations, we obtain the following stationary conditions:

$$\lambda'(\tau) = 0, \tag{22a}$$

$$1 + \lambda(\tau)|_{\tau=t} = 0. \tag{22b}$$

Eq. (22a) is called Lagrange–Euler equation and Eq. (22b) is a natural boundary condition.

The Lagrange multiplier can therefore, be identified as $\lambda = -1$ and the variational iteration formula is obtained in the form of:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{\dot{u}_n + u_n u_n' + a u_n'' + b u_n''' + c u_n^{(4x)}\right\} \mathrm{d}\tau. \tag{23}$$

We start with the initial approximation of $u(x, 0)$ given by Eq. (20). Using the above iteration formula (23), we can directly obtain the other components as follows:

$$u_0(x,t) = kx^4, \tag{24}$$

$$u_1(x,t) = kx^4 - 4k^2x^7t - 12akx^2t - 24bkxt - 24ckt, \tag{25}$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left\{\dot{u}_1 + u_1 u_1' + a u_1'' + b u_1''' + c u_1^{(4x)}\right\} \mathrm{d}\tau. \tag{26}$$

5.2. ADM implement for fractional KdV–Burgers–Kuramoto equation with $\alpha = 1$ (Eq. (18))

Now let us consider the application of ADM for Eq. (18). If Eq. (18) is dealt with this method, it is formed as

$$L_t u(x,t) = -(uL_x u + aL_{xx}u + bL_{3x}u + cL_{4x}u), \tag{27}$$

where

$$L_t = \frac{\partial}{\partial t}, \qquad L_x = \frac{\partial}{\partial x}, \qquad L_{xx} = \frac{\partial^2}{\partial x^2}, \qquad L_{3x} = \frac{\partial^3}{\partial x^3}, \ldots. \tag{28}$$

If the invertible operator $L_t^{-1} = \int_0^t (\cdot)\,\mathrm{d}t$ is applied to Eq. (27), then

$$L_t^{-1}L_t u(x,t) = -L_t^{-1}(uL_x u + aL_{xx}u + bL_{3x}u + cL_{4x}u), \tag{29}$$

is obtained. By this

$$u(x,t) = u(x,0) - L_t^{-1}(uL_x u + aL_{xx}u + bL_{3x}u + cL_{4x}u). \tag{30}$$

Here the main point is that the solution of the decomposition method is in the form of

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{31}$$

Substituting from Eq. (31) in (30), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) \;=\; & u(x,0) - L_t^{-1}\left(\sum_{n=0}^{\infty} u_n(x,t).L_x \sum_{n=0}^{\infty} u_n(x,t) + aL_{xx}\sum_{n=0}^{\infty} u_n(x,t)\right. \\ & \left. +\; bL_{3x}\sum_{n=0}^{\infty} u_n(x,t) + cL_{4x}\sum_{n=0}^{\infty} u_n(x,t)\right).\end{aligned} \tag{32}$$

Thus according to Eq. (12) approximate solution can be obtained as follows:

$$u_0(x,t) = kx^4, \tag{33}$$

$$u_1(x,t) = -4k^2x^7t - 12akx^2t - 24bkxt - 24ckt, \tag{34}$$

$$u_2(x,t) = -\int_0^t (uL_x u + aL_{xx}u + bL_{3x}u + cL_{4x}u)\mathrm{d}t. \tag{35}$$

The approximate solution of Eq. (18) is obtained as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t). \tag{36}$$

By the drawing of 3D figures of VIM solution and ADM solution, we can see that those figures are similar to each other. The results obtained from VIM and ADM methods are in excellent agreement with results of HAM method in Ref. [27].

5.3. VIM implement for fractional KdV–Burgers–Kuramoto equation with $\alpha = 0.5$ (Eq. (19))

Now we consider the application of VIM to Eq. (19) with the initial condition (20).

Its correction variational functional in x and t can be expressed, respectively, as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{\dot{u}_n + u_n u_n^{(0.5x)} + a u_n'' + b u_n''' + c u_n^{(4x)}\right\} \mathrm{d}\tau, \tag{37}$$

where prime indicates a differential with respect to x and dot denotes a differential with respect to t, λ is general Lagrangian multiplier.

After some calculations, we obtain the following stationary conditions:

$$\lambda'(\tau) = 0, \tag{38a}$$

$$1 + \lambda(\tau)|_{\tau=t} = 0. \tag{38b}$$

Eq. (38a) is called Lagrange–Euler equation and Eq. (38b) is a natural boundary condition.

The Lagrange multiplier can therefore, be identified as $\lambda = -1$ and the variational iteration formula is obtained in the form of:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{\dot{u}_n + u_n u_n^{(0.5x)} + a u_n'' + b u_n''' + c u_n^{(4x)}\right\} \mathrm{d}\tau. \tag{39}$$

Again we start with the initial approximation of $u(x,0)$ given by Eq. (20). Using the above iteration formula (39), we can directly obtain the other components as follows:

$$u_0(x,t) = kx^4, \tag{40}$$

$$u_1(x,t) = \frac{1}{35}\,\frac{k(35x^4\sqrt{\pi} - 128kx^{(\frac{15}{2})}t - 420ax^2t\sqrt{\pi} - 840bxt\sqrt{\pi} - 840ct\sqrt{\pi})}{\sqrt{\pi}}, \tag{41}$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left\{\dot{u}_1 + u_1 u_1^{(0.5x)} + a u_1'' + b u_1''' + c u_1^{(4x)}\right\} \mathrm{d}\tau. \tag{42}$$

5.4. ADM implement for fractional KdV–Burgers–Kuramoto equation with $\alpha = 0.5$ (Eq. (19))

Eq. (19) can be rewritten in operator form as

$$L_t u(x,t) = -(u(x,t) D_x^{0.5} u(x,t) + aL_{xx}u(x,t) + bL_{3x}u(x,t) + cL_{4x}u(x,t)), \tag{43}$$

where

$$L_t = \frac{\partial}{\partial t}, \qquad L_x = \frac{\partial}{\partial x}, \qquad L_{xx} = \frac{\partial^2}{\partial x^2}, \qquad L_{3x} = \frac{\partial^3}{\partial x^3}, \ldots \tag{44}$$

and $D_x^{0.5}$ is the Riemann–Liouville derivative of order 0.5.

If the invertible operator $L_t^{-1} = \int_0^t (\cdot)\,\mathrm{d}t$ is applied to Eq. (43), then

$$L_t^{-1} L_t u(x,t) = -L_t^{-1}(u(x,t) D_x^{0.5} u(x,t) + aL_{xx}u(x,t) + bL_{3x}u(x,t) + cL_{4x}u(x,t)), \tag{45}$$

is obtained. By this

$$u(x,t) = u(x,0) - L_t^{-1}(u(x,t) D_x^{0.5} u(x,t) + aL_{xx}u(x,t) + bL_{3x}u(x,t) + cL_{4x}u(x,t)), \tag{46}$$

is found. Here the main point is that the solution of the decomposition method is in the form of

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{47}$$

Substituting from Eq. (47) in (46), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) \;=\; & u(x,0) - L_t^{-1}\left(\sum_{n=0}^{\infty} u_n(x,t) D_x^{0.5} \sum_{n=0}^{\infty} u_n(x,t) + aL_{xx} \sum_{n=0}^{\infty} u_n(x,t)\right. \\ & \left. +\; bL_{3x} \sum_{n=0}^{\infty} u_n(x,t) + cL_{4x} \sum_{n=0}^{\infty} u_n(x,t)\right), \end{aligned} \tag{48}$$

is found.

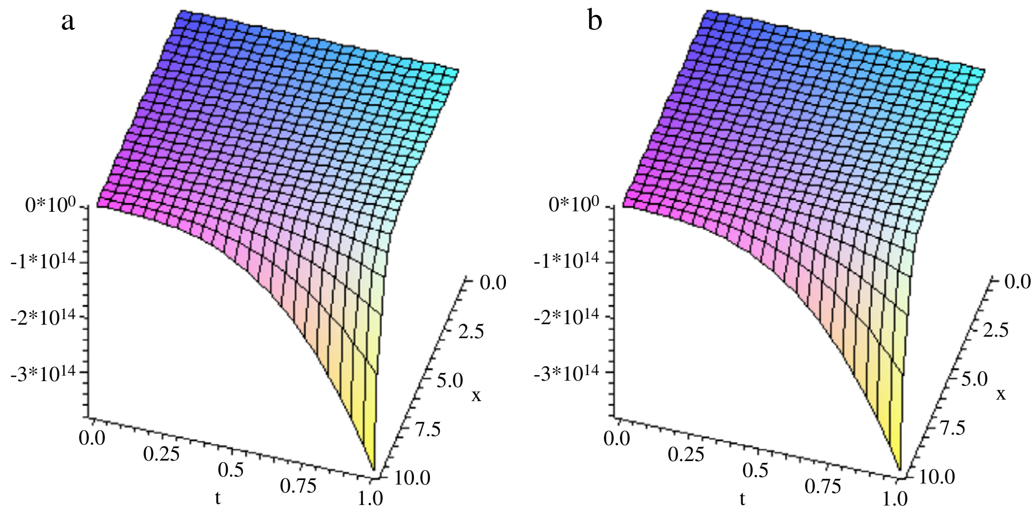

Fig. 1. For the fractional KdV–Burgers–Kuramoto equation with the initial condition (20) of Eq. (18) for $a = b = c = k = 1$, VIM result for $u(x, t)$ is, respectively (1a) and ADM solution (1b).

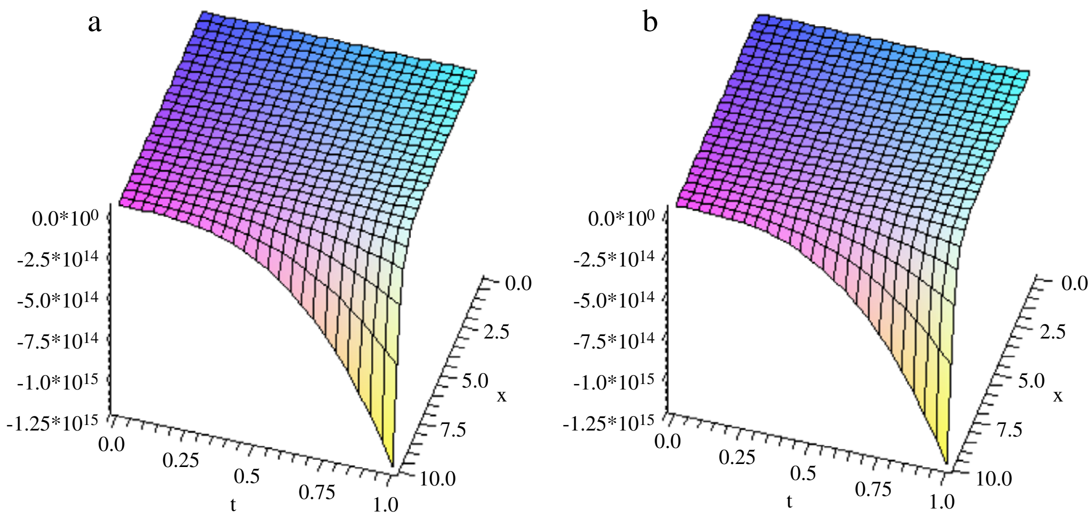

Fig. 2. For the fractional KdV–Burgers–Kuramoto equation with the initial condition (20) of Eq. (19), for a = b = c = k = 1, VIM result for $u(x, t)$ is, respectively (2a) and ADM solution (2b).

Thus according to Eq. (12) approximate solution can be obtained as follows:

$$u_0(x,t) = kx^4, \tag{49}$$

$$u_1(x,t) = \frac{-4}{35}\,\frac{kt(32kx^{(\frac{15}{2})} + 105ax^2\sqrt{\pi} + 210bx\sqrt{\pi} + 210c\sqrt{\pi})}{\sqrt{\pi}}, \tag{50}$$

$$u_2(x,t) = -\int_0^t (u(x,t)D_x^{0.5}u(x,t) + aL_{xx}u(x,t) + bL_{3x}u(x,t) + cL_{4x}u(x,t))\mathrm{d}t. \tag{51}$$

The approximate solution of Eq. (19) is obtained as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t). \tag{52}$$

By the drawing of 3D figures of VIM solution and ADM solution, we can see that those figures are similar to each other.

Similarly the results obtained from VIM and ADM methods are in excellent agreement with results HAM method in Ref. [27].

Fig. 1 shows the fractional KdV–Burgers–Kuramoto equation with the initial condition (20) of Eq. (18) for $a = b = c = k = 1$, VIM result for $u(x, t)$ is, respectively (1a) and ADM is solution (1b). And Fig. 2 illustrates the fractional KdV–Burgers–Kuramoto equation with the initial condition (20) of Eq. (19), for $a = b = c = k = 1$, VIM result for $u(x, t)$ is, respectively (2a) and ADM is (2b).

6. Conclusion

Many powerful methods have been presented to the study of explicit solutions to nonlinear integer-order differential equation [5–30]. In this paper, He's variational iteration method and Adomian's decomposition method has been successfully applied to find the solution of space-fractional diffusion equation. All cases show that the results of the VIM method are in excellent agreement with those of ADM solutions and the obtained solutions are shown graphically. In addition, the results obtained from these two methods are in excellent agreement with HAM method results in Ref. [27]. In our work; we use the Maple Package to calculate the functions obtained from the variational iteration method and Adomian decomposition method.

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